INTRODUCTION IMPORTANCE OF PROPOSED INVESTIGATION, OBJECTIVE

Fractional Calculus: The word Fractional Calculus has been running for four Hundred years ago. The concept of fractional calculus is nearly as old as the classical calculus. It is a mathematical approach dealing with the derivatives and integrals of non-integer order. For a long time, fractional calculus was developed mainly as a purely theoretical field of mathematics without its real applications. But in recent decades, it has been found that fractional calculus is very useful from both theoretical as well as applied point of views. Due to nonlocal property and memory effect of fractional operators, it is giving rise to numerous new and important applications across various fields of science and engineering.

Leibniz invented the notation $\frac{d^n y}{dx^n}$. A correspondence between Leibniz and L'Hopital in the mid 17th century considered the idea of derivative of order one half. Since then, many mathematicians have worked on the concept of differentiation and integration of non-integer order. Several mathematicians namely Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov,Weierstrass, Heaviside, Mittag-Leffler, Feller, Erdelyi and Riesz etc. have made significant contributions in the field of fractional calculus.

In 1812, Laplace defined a fractional derivative by means of an integral. Later, Lacroix, devoted two of 700-page of his text to this topic, mentioning about the derivative of arbitrary order. Lacroix easily developed the nth derivative of $y = x^m$ as

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, m \geq n, m, n \in N$$

Using Legendre's symbol Γ , he wrote the above equation as:

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

By replacing n with 1/2 and taking m = 1, it yields

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = \frac{2}{\sqrt{\pi}} \sqrt{x}$$

The first successful attempt of Fractional Operations was made by **Niels-Henrik Abel** in 1843.

Abel integral equation is in the form as

$$K = \int_0^x (x-t)^{-\frac{1}{2}} f(t)dt$$

In 1832 Liouville and in 1848 C. J. Hargreave gave generalization of Leibniz's n[th] derivative of a product when n is not a positive integer.

$$D^{\upsilon} f(x)g(x) = \sum_{n=0}^{\infty} \binom{\upsilon}{n} D^n f(x) D^{\upsilon-n} g(x)$$

Here D^{υ} is the Differentiation Operator of order υ, $D^{\upsilon-n}$ is a Fractional Operator, and $\binom{\upsilon}{n}$ has usual meaning.

Although, this rule is valid for υ is a positive integer but modern mathematicians would seek to justify this rule when υ is arbitrary.

Lacroix's method offered no clue as to possible application, but the result is same as that yielded by the present day Riemann-Liouville definition of fractional derivative **Riemann- Liouville** saw the paper written by N.Ya .Sonin [1869]. His starting point was Cauchy's Integral Formula. The n^{th} derivative of Cauchy's Integral Formula is given by

$$D^n f(Z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - Z)^{n+1}} d\xi$$

The method of contour integration produced the definition

$${}_C D_x^{-\upsilon} f(x) = \frac{1}{\Gamma(\upsilon)} \int_c^x (x-t)^{\upsilon-1} f(t)dt \,, Re(\upsilon) > 0$$

for integration to an arbitrary order.

The important version occurs when c=0,

$${}_0 D_x^{-\upsilon} f(x) = \frac{1}{\Gamma(\upsilon)} \int_0^x (x-t)^{\upsilon-1} f(t)dt \,, Re(\upsilon) > 0$$

This form is known as the Riemann-Liouville Fractional Integral.

The first successful study about developing the Fractional different integral theory for complex variable functions belongs to the Sonin and Letnikov. They give us Riemann's-Liouville definition of integration of order α.

$$ {}_0D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} f(t) dt \qquad \ldots (1) $$

The following two equations define by Riemann-Liouville Caputo Fractional Derivative of order α respectively:

$$ D^{\alpha} f(x) = \frac{d^m}{dx^m} \left(I^{m-\alpha} f(x) \right) $$

$$ D^{\alpha} f(x) = I^{m-\alpha} \left(\frac{d^m}{dx^m} f(x) \right) $$

The Fractional derivative of order α is defined as

$$ {}_0D_z^{\alpha} f(z) = \frac{d}{dz} \left[\int_0^z \frac{(z-t)^{\alpha-1}}{\Gamma(1-\alpha)} f(t) dt \right] \qquad \ldots (2) $$

The general integral operators (1) and (2) were introduced by kober and many mathematicians in various forms.

The generalization of (1) and (2) have applied in hyper geometric function by Riemann-and Liouville.

Univalent Function Theory: Univalent Function theory and certain subclasses of it is defined geometric conditions so a class of univalent function theory is an important branch of the geometric function theory.

Definition of Univalent Function: A single valued function $f(z)$ is said to be univalent in a domain D which is subset of the complex plane$\mathbb{C}$, if it never takes the same value twice i.e.

$$ f(z_1) = f(z_2) \quad \forall\, z_1, z_2 \in D \text{ with } z_1 = z_2 $$

or, $$ f(z_1) \neq f(z_2) \quad \forall\, z_1, z_2 \in D \text{ with } z_1 \neq z_2 $$

In other words we can say that the function $f(z)$ is one-one in the domain D.

Let $f(z)$ is analytic and univalent function in the unit disk $D = \{z: |z| < 1\}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus $f \in \sum$ is the class of univalent functions defined by geometric conditions.

Let $\sum$ be the class of functions $f(z)$ normalized by

$$ f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad , |z| < 1 $$

or $$f(z) = z + \sum_{m=2}^{\infty} a_m z^m$$

which is the analytic in the unit disk$D = \{z: z \in \mathbb{C} \ \& \ |z| < 1\}$.

Let S denote the subclass of $\sum$ consisting of function which are univalent.

In 1916, Bieberbach states that if $f(z) \in S$, then $|a_2| \leq 2$ with equality if and only if $f(z)$ is a rotation of the Koebe function or mathematically states that$f(z) \in S \Longrightarrow |a_2| \leq 2$.

In 1984 De-Branges generalized this result with proof that$f(z) \in S \Longrightarrow |a_2| \leq n, n = 2,3, \ldots$. Also Koebes One-quarter theorem states that: The range of every function $f(z)$ of class S contains the disk$|\omega| \leq \frac{1}{4}$.

Hadamard Product or Convolution: If $f(z)$and $g(z)$ are functions belonging to the class $\sum$ and defined by the

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

and $$g(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

Then $$(f * g)(z) = z + \sum_{n=2}^{\infty} b_n c_n z^n = (g * f)(z)$$

is known as the Hadamard Product or Convolution of the $f(z)$ and $g(z)$ function.

The concept of Hadamard Product considerably given new direction to certain interesting results in the Geometric Function Theory.

The importance of this subject or Fractional Calculus further lies in the fact that during the last many years members of international conferences dedicated exclusively to Fractional Calculus and its applications were held. Some are:

1. NIOCA Conference, Szczecin, Poland, 2015.
2. One-day Conference in Mathematics and Fractional Calculus, Alexandria University, 2015.
3. IC FDA conference, Catania, Italy, 2014.
4. Conference on Fractional calculus and its applications in Nihon University, Tokyo, Japan, 1989.
5. Conference on Fractional calculus and its applications in University of New Haven, United States, 1974.

The Upcoming conference on the Fractional Calculus will be held in **Novi Sad, Serbia**, in July, 2016 titled as **"International Conference on fractional Differentiation and its Applications"**.

Fractional Calculus has a great importance in applied science. During the last two decades this function has come to prominence. The Fractional Calculus also used in pure Mathematics and it has changed in some particular cases to meet the requirement of physical reality and objectives. It is also used in fluid dynamics and has a great importance in many fields as physical, biological, engineering, statistic, medical, earth sciences and many more.

Objectives: Objectives make the study easy to reach the conclusion of the investigation undertaken.

The main objectives of the present investigations shall be focused in the field of Fractional Calculus Operators on several new classes of analytical functions and derive new results.

1. To derive useful characteristics properties of meromorphic functions.
2. To investigate inclusion properties for certain subclasses of analytic function and meromorphic function.
3. To construct unified class of univalent/multivalent functions and their geometric properties.
4. Applications of Fractional Calculus Operators in various fields.
5. To analyze the growth and distortion theorems of multivalent functions

REVIEW (work already done on the subject of proposed research study)

A review of literature of recent developments on the topic of fractional calculus operators is given below.

Fractional calculus is based on several definitions for the operators of integration and differentiations of arbitrary order. Among the various definitions of fractional derivatives, the Grunwald, Riemann-Liouville and Caputo's fractional derivatives are widely used in literature. Generally, the operators of integration and differentiation are interchanged in the corresponding definitions of Caputo fractional derivative and Riemann-Liouville fractional derivative. Fractional differential equations with Caputo fractional derivative allow traditional initial and boundary conditions. However, the fractional differential equations based on the Riemann-Liouville fractional derivative leads to a conflict of interest between the well-established mathematical theory, because it allows the initial conditions in terms of fractional integrals and their derivatives.

Besides the Riemann-Liouville definition of fractional operators, there are several modifications and their generalizations that have been studied. Kiryakova presented a brief history about the operators of the generalized fractional calculus. In mathematical texts and applications, Hadamard, Erdelyi-Kober, Saigo, multi-Erdelyi-Kober operators are often used. The authors have contributed in the development of the theory of a hyper-geometric operator.

Over the last few years, a deep understanding of the behavior of various complex systems and the generalization of fractional differential operators were subjected to intense debates. But whether the generalized fractional derivatives contain physically interpretable initial conditions, is the fractional derivative of a constant 0? Keeping the concept of Caputo's fractional derivative, recently several authors have introduced Caputo-type derivative on generalized fractional operators.

The Riemann-Liouville Fractional integral operator of order α and its generalization has been studied by many mathematician time to time for long time.

It is well known that the Riemann-Liouville fractional derivative of a constant is not zero. A useful alternative for the Riemann-Liouville fractional derivative is the Caputo's fractional derivative. But the definition of Caputo's derivative would be defined for the differentiable function only. Another fact on this derivative is that, if anyone wants to find the derivative, it will require its second derivative of a function. Recently, Jumarie proposed a modification of the Riemann-Liouville definition by means of fractional difference for non-differentiable function.

The development of the subject appears in Liu, J. L. and Srivastava, H. M. [19], Miller, S.S and Ross, B. [23], Oldhalm, K.B. and Spanier J. [27]. Recently in mathematical analysis there are various fields where Fractional Calculus Operators have been effectively used by many investigators including Bapna,I.B., Mathur Nisha [5], [6], [7], El-Ashwah, R. M [12], Goodman, A. W.[13], Hayman, W. K.[14], Prajapat, J.K. and Raina R.K. [33], and many more.

The rudiments of the analytic function theory had already emerged at the beginning of the 20^{th} century in the investigation of Koebe in 1907, Gromwells proof of the area theorem in the 1914 and Bieberbach's estimate for the second coefficient of a normalized univalent function in 1916.

However, the full development of the subject was possible after 1945 as a result of the fundamental researches of Bernardi, S.D.[8], Duren P.L.[9], [10], Libera, R.J.[17], Littlewood, J.E[18], Padmanabham, K.S. Parvatham, R.[29], Ponnusamy, S[32], Robertson, M.S[37], Silverman, H[39] and many others. Their works are also reported in the books of Duren [9], [10], Goodman [13], Miller and Mocano [21], [22], and Srivastava and Owa [40], [41].

At a present time many investigators including Aouf, M.K [2], Bapna S. and Nisha Mathur [5], [6], [7], Patel,J.[31], Poonusamy, S. [32], Prajapat, J.K. and Raina R.K. [33], Raina R.K. and Bapna S.[35], [36], Srivastava, H.M.[40], and many more are engaged in the field and have given many properties of geometric function theory with the help of Fractional Calculus, differential subordinate technique and concept of Hadamard Product (Convolution structure).

RESEARCH GAPS INDENTIFIED IN THE PROPOSED FIELD OF INVESTIGATION (BASED ON THE REVIEW):

Fractional Calculus was born in 1695 when a correspondence between Leibniz and L'Hopital in the mid 17^{th} century considered the derivative of order one half, while initial stage of foundation of Fractional Calculus Operators was very slow progress made in its theory and application due to lack of support from physics and mechanics, especially due to a conflict with Newtonian integer order system of classical physics. In 1812, Laplace defined a fractional derivative.

The rudiments of the analytic function theory had already emerged at the beginning of the 20^{th} century in the investigation of Koebe in 1907, Gromwells 1914, and Bieberbach 1916. However, the full development of the subject was possible after 1945 as a result of the fundamental researches of Bernardi, Duren, Robinson, Spacek and many others.

The year 1974, saw the first International Conference on Fractional Calculus, held at the University of New Haven. The Proceedings of the conference were published by Springer verlog [Ross, 1975]. In India, Raina, R.K. and Saxena R.K. have produced many papers; in Canada, Srivastava H.M.; in Venezuela Kalla S. have all become well known for these contributions to the Fractional Calculus.

In period 1985-2000 a modest amount of published work appeared on the subject of the Fractional Calculus Operators most recent works are in this field have Srivastava H.M., Prajapat J.K., Raina R.K., Bapna I.B., Mathur Nisha and many others. with all these mentioned above, in 2000, Kleinz and Osler published an introductory article named "A child's garden of fractional derovatives".

MAJOR HYPOTHESIS:

Any research study expressed as it should be logical and scientifically legal. Practically proved scientific study is very useful. A scientific enquiry starts with the identification of problem in a logician's point of view. When necessary facts are unknown then the results must shows the correct knowledge by manipulations, guesses or supposing the fact can destruct the real correct result of the research study.

Derivation in useful directions of fractional calculus and evaluation of some investigations with the help of this could be served as well. We formed a hypothesis on its important necessity and applications. In the scientific field of science and social sciences the role of Fractional Calculus is very useful in today's scenario. We will formulate a hypothesis on this utility basis; actually it guided me to take the study. By the formulation of our hypothesis we trying to form a definite point to several new classes of analytical and meromorphic functions with various results concerned with their subclasses.

Actually hypothesis is very helpful in collecting proper and useful required data instead of gathering unrequited data which can create some problem under study.

RESEARCH METHODOLOGY:

In the starting we shall consider some subclasses of analytic function and meromorphic function in the unit disk D such that $D = \{z: |z| < 1\}$, and after we shall apply the Riemann-Liouville Fractional Calculus Operators and various types of operators to obtain new results.

Many researchers will be based upon the concept of Hadamard product, Schwarz lemma. Many also are included in general way. Some various may also be used according to the requirement of our problem as data, books, journals, thesis, laptops, software (math-type, math-tools, math-cad, and latex) and many more.

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